Lesson 3: Permutations with Repeating (Identical) Objects

Consider the number of 3-letter arrangements possible using the letters of the word *off.* For a regular 3-letter word, we would assume that there are 3! Or 6 arrangements of the letters. However, when we list the arrangements, there are only 3 distinct arrangements:

fof ffo

off *fof* *ffo*

1 2 3

This is because of the repeated f's. When we have a set of n objects when a of some kind are identical, b of some kind are identical, c of some kind are identical, etc., the number of distinct arrangements can be found by $\boxed{\frac{n!}{a!b!c!\ldots}}$ ← arranging n object where some are identical.

Example 1:

In how many distinct ways can you rearrange the letters in the word FLUFFY?

$3! = 3 \cdot 2 \cdot 1 = 6$

$1! = 1$

$$\frac{6!}{3!1!1!1!} = \frac{720}{6} = 120 \text{ ways.}$$

$\left.\begin{array}{l} F's = 3 \\ U's = 1 \\ L's = 1 \\ Y's = 1 \end{array}\right\}$ total of 6

Example 2: In how many ways can you re-arrange the letters in the word MISSISSIPPI?

$$\frac{11!}{1!4!4!2!} = 34,650 \text{ ways}$$

$\left.\begin{array}{l} M's = 1 \\ I's = 4 \\ S's = 4 \\ P's = 2 \end{array}\right\}$ total of 11

Example 3: How many *different* six-digit numerals can be written using all of the following six digits: 4, 4, 5, 5, 5, 7?

$$\frac{6!}{2!3!1!} = \frac{6 \cdot 5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{2} \cdot 1)(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})(1)} = 60 \text{ ways.}$$

Example 4: How many distinct arrangements are there using the letters in the word EFFERVESCENCE if:

E's = 5
F's = 2
R's = 1
V's = 1
S's = 1
C's = 2
N's = 1
} 13 letters

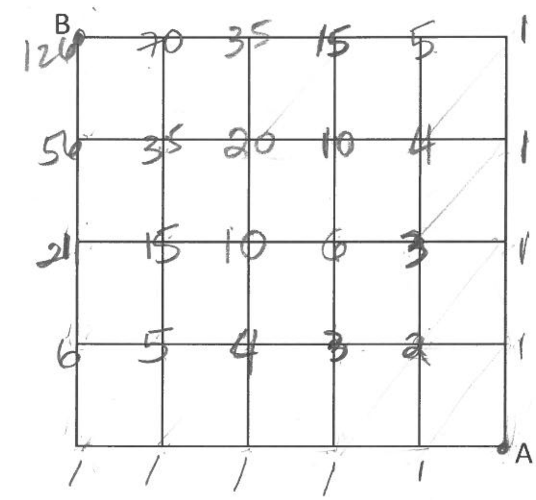
a) there are no restrictions.

$$\frac{13!}{5!2!1!1!1!2!1!} = \boxed{12,972,960}$$

b) the rearrangement must start with the letter 'V'.

$$\frac{1}{V} \times \frac{12!}{5!2!1!1!1!2!1!} = \boxed{997,920}$$

Example 5: How many paths can you follow from A to B in the grid below if you move only up or to the left?

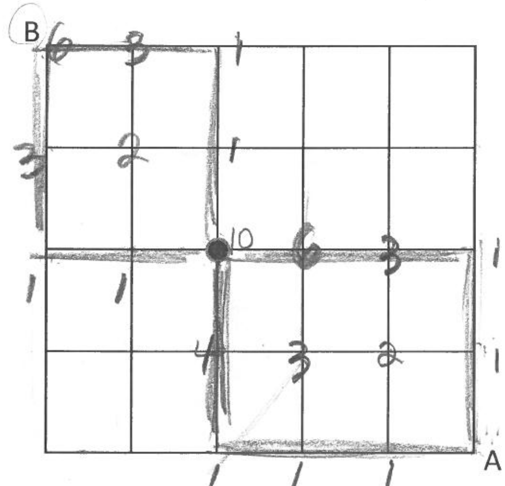


$$\frac{{}_9P_9}{{}_5P_5 \times {}_4P_4} = \frac{9!}{5!\,4!}$$

$$= \boxed{126 \text{ ways}}$$

↑ lefts ↑ ups

b) How many paths can you follow from A to B in the grid below if you move only up or to the left and you must pass through the point?



From A to Pt. From Pt to B

10 × 6

= 60 ways

Assignment Time! Work on p.712- 3 – 9, 12, MC 1-3 arrangement of objects

Lesson 4: Permutations with Restrictions

Sometimes a permutation problem will involve some restrictions. Whenever you encounter a problem with constraints or restrictions, always address the choices for the restricted positions first.

Example 1: How many permutations of the letters of the word ORANGE are there if they must begin with a vowel?

all characters are distinct. 6

$\underset{\text{Vowel}}{_3P_1} \times \underset{\text{rest of letters}}{_5P_5}$ | $\underset{\text{Vowel}}{\underline{3}} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 360$ ways

How many permutations for the word ORANGE are there that begin AND end with a vowel?

$\underset{\text{vowel}}{\underline{3}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underset{\text{vowel}}{\underline{2}} = 144$ ways.

Example 2: How many ways can 3 men and 7 women be seated in a row if there has to be a woman at the beginning and end of the row?

$\underset{\text{woman}}{\underline{7}} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underset{\text{woman}}{\underline{6}} = \boxed{1,693,440 \text{ ways}}$

Example 3: How many was can the letters of ELATION be arranged if

a) there are no restrictions? $\underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 7! = 5040$

b) consonants and vowels must alternate?

$\underset{V}{\underline{4}}\ \underset{C}{\underline{3}}\ \underset{V}{\underline{3}}\ \underset{C}{\underline{2}}\ \underset{V}{\underline{2}}\ \underset{C}{\underline{1}}\ \underset{V}{\underline{1}} = 144$ ways

c) a consonant must be in the middle of each arrangement?

$\underline{6}\ \underline{5}\ \underline{4}\ \underset{C}{\underline{3}}\ \underline{3}\ \underline{2}\ \underline{1} = 2160$ ways

Permutations Involving Groups

Example 1: Five people (A, B, C, D, E) are seated on a bench. How many ways can they be arranged if:

a) there are no restrictions?

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = {}_5P_5 = 120$

b) E must sit in the middle?

$\underset{}{4} \cdot 3 \cdot \underset{E}{1} \cdot 2 \cdot 1 = 24$ ways.

c) A and B must sit together?

A and B treat as 1 object

$4!\,2! = (4 \times 3 \cdot 2 \cdot 1)(2 \cdot 1) = 48$ ways

d) d) A and B cannot be together?

$120 - 48 = 72$ ways

Example 2: How many ways can four girls and three boys be arranged in a row if:

a) A boy must be at each end of the row.

$\underset{Boy}{3} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \underset{Boy}{2} = 720$ ways

b) The boys must be together.

4 Girls and a group of boys $\rightarrow 5!\,3! = 720$ ways.

c) The girls must be together.

$4!\,4! = 576$ ways

d) The girls must be together and the boys must be together.

$2!\,3!\,4! = 288$ ways.

Example 3: How many ways can all the letters of ***audio*** be arranged if

a) each arrangement must begin with a vowel and end with a consonant?

$\underset{Vowel}{4}\ 3\ 2\ 1\ \underset{Consonant}{1} = 24$

b) the vowels must be together?

$2!\,4! = 48$ ways.

c) the vowels cannot be together?

Let's figure all possibilities w/o restrictions.

$5! = 120$

$120 - 48 = 72$ ways. //