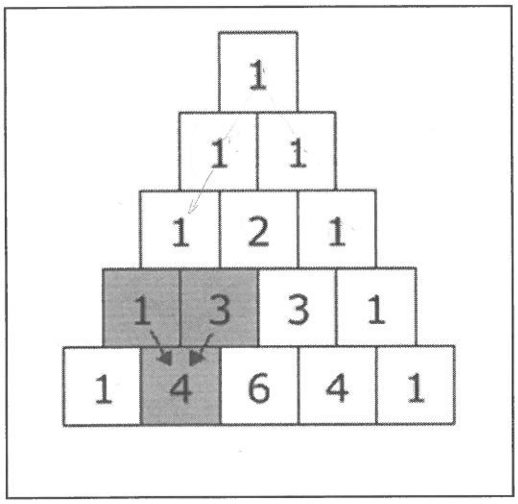
Lesson 6: Pascal's Triangle

Pascal's Triangle: One of the most interesting number patterns is Pascal's Triangle. The triangle is built by starting with a "1" at the top and then continuing to place numbers below it in a triangular pattern. Each number (except for the starting "1") is the sum of the numbers directly above it.



1 Row 1

1 1 Row 2

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1 Row 11

1 11 55 165 330 462 462 330 165 55 11 1 Row 12

Binomial expansion refers to the expanding of a binomial. For example, the following three binomials have been shown in their expanded form.

$$(x+1)^2 = 1x^2 + 2x + 1$$
$$(x+1)^3 = 1x^3 + 3x^2 + 3x + 1$$
$$(x+1)^4 = 1x^4 + 4x^3 + 6x^2 + 4x + 1$$

There are quicker ways to expand these binomials that doesn't involve multiplying out and collecting like terms. Knowing about Pascal's Triangle or the Binomial Theorem can help us to expand binomials quickly.

\# of terms

1 ← $(x+1)^0$ 1

1 1 ← $(x+1)^1$ 2

1 2 1 ← $(x+1)^2$ 3

1 3 3 1 ← $(x+1)^3$ 4

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

$(x+1)^n$ $n+1$ terms

The table below shows how Pascal's Triangle relates to the coefficients of a binomial expansion. Notice that for binomial $(x+y)^n$, the coefficients match the entries in the $(n+1)^{st}$ row of the triangle.

Binomial Expansion	Row #	Entries in Pascal's Triangle
$(x+1)^2 = 1x^2 + 2x + 1$	3	1, 2, 1
$(x+1)^3 = 1x^3 + 3x^2 + 3x + 1$	4	1, 3, 3, 1
$(x+1)^4 = 1x^4 + 4x^3 + 6x^2 + 4x + 1$	5	1, 4, 6, 4, 1
etc.		

The formula ${}_nC_r$ or $\binom{n}{r}$ can be used to determine entries in Pascal's triangle.

Example 1:

$(x+1)^5$ a) Determine the entry in Row 6, term 5 on Pascal's Triangle. term 5 = 5

1 5 10 10 $\boxed{5}$ 1

$(x+1)^4$ b) Determine the entry in Row 5, term 3 on Pascal's Triangle. term 3 = 6

1 4 $\boxed{6}$ 4 1

c) Determine the entries in the seventh row of Pascal's Triangle. $(x+1)^6$

1 6 15 20 15 6 1

d) Determine the entries in the fifth row of Pascal's Triangle.

1 4 6 4 1

Assignment Time! Work on p.737- 1 – 4

Lesson 7: The Binomial Theorem

The binomial theorem can be used to expand binomials of the form $(x + y)^n$.
Note the following:

- The expansion of $(x + y)^n$ contains $(n + 1)$ terms. That is, the expansion contains one more term than the value of "n".
- The exponents on "x" start at n and decrease by 1 in each subsequent term until we are left with x^0 in the last term.
- The powers of 'y' start at y^0 in the first term, and increase to y^n in the last term.
- In each term the exponents on 'x' and 'y' add up to a total of n.
- The coefficients of each term form Pascal's Triangle or can be found using the combination formula.

Example 1: Use the Binomial Theorem to expand $(x-3)^3$ $n=3$ there will be 4 terms.

$$(x-3)^3 = {}_3C_0(x)^3(-3)^0 + {}_3C_1(x)^2(-3)^1 + {}_3C_2(x)^1(-3)^2 + {}_3C_3(x)^0(-3)^3$$
$$= 1(x^3)(1) + 3x^2(-3) + 3(x)(+9) + 1(1)(-27)$$
$$= x^3 - 9x^2 + 27x - 27$$

Example 2: Use the Binomial Theorem to expand $(2a+3)^4$ $n=4$ 5 terms.

$$(2a+3)^4 = {}_4C_0(2a)^4(3)^0 + {}_4C_1(2a)^3(3)^1 + {}_4C_2(2a)^2(3)^2 + {}_4C_3(2a)^1(3)^3 + {}_4C_4(2a)^0(3)^4$$
$$= 1(16a^4)(1) + 4(8a^3)(3) + 6(4a^2)(9) + 4(2a)(27) + 1(1)(81)$$
$$= 16a^4 + 96a^3 + 216a^2 + 216a + 81$$

Example 3: Determine the first three terms of $(2x-y)^{12}$ $n=12$ $12+1=13$ terms.

$${}_{12}C_0(2x)^{12}(-y)^0 + {}_{12}C_1(2x)^{11}(-y)^1 + {}_{12}C_2(2x)^{10}(-y)^2$$
$$= 1(4096x^{12})(1) + 12(2048x^{11})(-y) + 66(1024x^{10})(y^2)$$
$$= 4096x^{12} - 24576x^{11}y^1 + 67584x^{10}y^2$$

Example 4: Use the Binomial Theorem to expand $(3x-y)^5$ $n=5$ $n+1$ $5+1=6$ terms.

$${}_5C_0(3x)^5(-y)^0 + {}_5C_1(3x)^4(-y)^1 + {}_5C_2(3x)^3(-y)^2 + {}_5C_3(3x)^2(-y)^3 + {}_5C_4(3x)^1(-y)^4 + {}_5C_5(3x)^0(-y)^5$$

$$= 1(243x^5) + 5(81x^4)(-y) + 10(27x^3)(y^2) + 10(9x^2)(-y^3) + 5(3x)(y^4) + 1(1)(-y^5)$$

$$= \boxed{243x^5 - 405x^4y + 270x^3y^2 - 90x^2y^3 + 15xy^4 - y^5}$$

Example 5: Determining a Specific Term in an Expansion

The general term, or the formula to find any term in a binomial expansion, $(x+y)^n$ is:

$$\boxed{t_{k+1} = {}_nC_k x^{n-k} y^k} \leftarrow \text{Follow the formula.}$$

a) Determine the 6th term in the expansion of $(x-2)^{11}$ $\boxed{n=11}$

$t_6 = t_{k+1}$

$\boxed{k=5}$

$t_6 = t_{5+1}$

$$t_6 = {}_{11}C_5(x)^{11-5}(-2)^5$$

$$t_6 = 462(x^6)(-32)$$

$$t_6 = -14784x^6$$

b) Determine the 7th term in the expansion of $(2x-3y)^9$ $\boxed{n=9}$

$t_7 = t_{k+1}$

$k+1=7$

$\boxed{k=6}$

$$t_7 = {}_9C_6(2x)^{9-6}(-3y)^6$$

$$t_7 = 84(8x^3)(729y^6)$$

$$t_7 = 489{,}888x^3y^6$$

c) Determine the middle term in the expansion of $(x+5)^6$

$n=6$

There will be 7 terms

$t_4 = t_{k+1}$

$4 = k+1$

$\boxed{k=3}$

_ _ _ _ _ _ _

middle term

term 4.

$$t_4 = {}_6C_3(x)^{6-3}(5)^3$$

$$= 20(x^3)(125)$$

$$t_4 = 2500x^3$$

d) Determine the final term in the expansion of $\left(\frac{1}{x} - x^3\right)^{13}$

$n = 13$

$n+1$ terms

$13+1 = 14$ terms.

term #14

$t_{14} = t_{k+1}$

$k = 13$

$$t_{14} = {}_{13}C_{13}\left(\frac{1}{x}\right)^{13-13}\left(-x^3\right)^{13}$$

$$= 1(1)(-x^{39})$$

$$t_{14} = -x^{39}$$

e) In the expansion of $(a^3 - \frac{2}{a})^7$ which term, in simplified form, contains a^5?

write 3 terms, look for pattern.

$${}_7C_0\left(a^3\right)^7\left(\frac{-2}{a}\right)^0 + {}_7C_1\left(a^3\right)^6\left(\frac{-2}{a}\right)^1 + {}_7C_2\left(a^3\right)^5\left(\frac{-2}{a}\right)^2$$

$$a^{21}(1) + a^{18}\left(\frac{-2}{a}\right) \qquad a^{15}\left(\frac{4}{a^2}\right)$$

$$a^{21} \qquad a^{17} \qquad a^{13}$$

exponents go down by 4.

a^{21}	a^{17}	a^{13}	a^9	a^5
t_1	t_2	t_3	t_4	t_5

$\boxed{\text{terms will have } a^5}$

Assignment Time! $\boxed{\text{Work on p.743 5, 7, 11, MC 1-3}}$