Lesson 7: Sum and Difference Identities

The sum and difference identities are identities that work with the sum and the difference of two angles in standard position. The new identities are listed below:

The Sum and Difference Identities

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Special Angles: $0°, 30°, 45°, 60°, 90°$, And their multiples.

One application of these new identities is to find exact values of trigonometric ratios of new (non-'special') angles.

0.965925..

Example 1: Determine the exact value of $\sin 75°$. ← How can we use the special angles to get 75°?

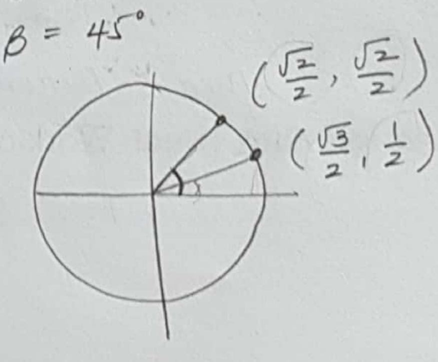
Use $30° + 45° = 75°$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha = 30°$

$\beta = 45°$

$$\sin 75 = \sin(30° + 45°)$$
$$= \sin 30 \cos 45 + \cos 30 \sin 45$$
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

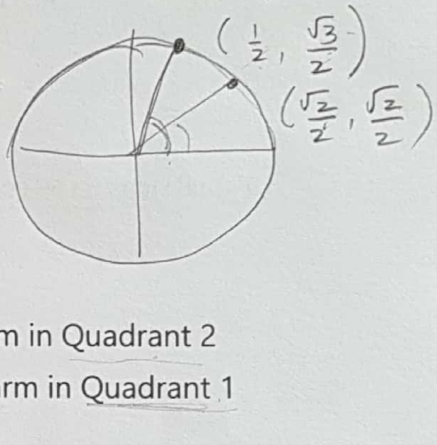


Special angles in Radian: $0^r, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.

Example 2: Determine the exact value of $\tan\frac{\pi}{12}$. $= 0.2679\ldots$

$\alpha = \frac{\pi}{3}$ $\quad \beta = \frac{\pi}{4}$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$



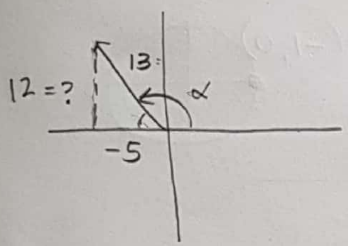
$$\tan\frac{\pi}{12} = \tan\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \left(\tan\frac{\pi}{3}\right)\left(\tan\frac{\pi}{4}\right)} = \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

Example 3: Given an angle α in standard position with its terminal arm in Quadrant 2 and $\cos\alpha = -\frac{5}{13}$, and angle β in standard position with its terminal arm in Quadrant 1 and $\sin\beta = \frac{3}{5}$, determine the exact value of $\cos(\alpha - \beta)$.



$\cos\alpha = \frac{-5}{13}$ ← adj, ← hyp

$\sin\alpha = \frac{12}{13}$ ← opp, ← hyp

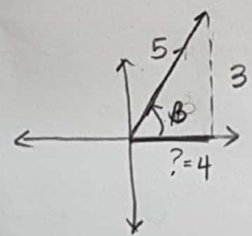
$x^2 + y^2 = r^2$

$(-5)^2 + y^2 = 13^2$

$y^2 = 169 - 25$

$\sqrt{y^2} = \sqrt{144}$

$y = 12$



$\sin\beta = \frac{3}{5}$ ← opp, ← hyp

$\cos\beta = \frac{4}{5}$ ← adj, ← hyp

$x^2 + y^2 = r^2$

$x^2 + 3^2 = 5^2$

$x^2 = 25 - 9$

$\sqrt{x^2} = \sqrt{16}$

$x = 4$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\frac{-5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$$

$$= \frac{-20}{65} + \frac{36}{65}$$

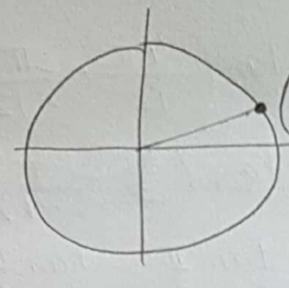
$$= \boxed{\frac{16}{65}}$$

Example 4: Write each expression in simplest form, then evaluate where possible.

a) $\dfrac{\tan\frac{\pi}{2} - \tan\frac{\pi}{3}}{1+\tan\frac{\pi}{2}\tan\frac{\pi}{3}}$ $= \tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \tan\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right)$

$= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$



$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

b) $\sin 8x \cos 3x - \cos 8x \sin 3x$ $= \sin(8x - 3x)$

$= \sin(5x)$

Example 5: Prove the following identities

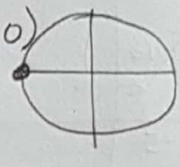
(0,1)

a) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

LS	RS
$\cos\left(\frac{\pi}{2} - x\right)$	
$\cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$	
$0(\cos x) + 1.(\sin x)$	
$0 + \sin x$	
$\sin x$	$\sin x$

LS = RS

b) $\sin(\pi - x) = \sin x$



(−1,0)

LS	RS
$\sin(\pi - x)$	
$\sin\pi\cos x - \cos\pi\sin x$	
$0(\cos x) - (-1)\sin x$	
$\sin x$	$\sin x$

LS = RS

VISUALIZE! Use desmos

$y = \cos\left(\frac{\pi}{2} - x\right)$

$y = \sin x$

You'll find that it produces the same graph. [Radian mode]

Example 4: Solve each of the following equations over the domain $[0, 2\pi)$.

a) $\sin 5x \cos 3x - \cos 5x \sin 3x = 1$

$\alpha \quad \beta \quad \alpha \quad \beta$

$\sin(5x - 3x) = 1$

$\sin^{-1} \sin 2x = \sin^{-1} 1$

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

CHECKING

$\sin\frac{5\pi}{4}\cos\frac{3\pi}{4} - \cos\frac{5\pi}{4}\sin\frac{3\pi}{4}$ | 1

$\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$\frac{2}{4} + \frac{2}{4}$

$\frac{4}{4}$

1

LS = RS

Since $b = 2$ the period is changed to

$p = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$

The solution is $x = \frac{\pi}{4}$

The general solution $\boxed{x = \frac{\pi}{4} + \pi k, \quad k \in \mathbb{Z}}$

b) $\cos 4x \cos x + \sin 4x \sin x = 1$

$\alpha \quad \beta \quad \alpha \quad \beta$

$\cos(4x - x) = 1$

$\cos^{-1} \cos(3x) = \cos^{-1} 1$

$\frac{3x}{3} = \frac{0}{3}$

$x = 0$

CHECKING:

$\cos 4(0) \cos 0 + \sin 4(0) \sin 0$ | 1

$\cos 0 \cos 0 + \sin 0 \sin 0$

$(1)(1) + (0)(0)$

$1 + 0$

1 | 1

LS | RS

Since $b = 3$ The period is changed to

$P = \frac{2\pi}{3}$

The solution is $x = 0$

The general solution $\boxed{x = 0 + \frac{2\pi}{3}k, \quad k \in \mathbb{Z}}$

Assignment Time! Work on: p.641- 4, 5, 7, 8, 9, 12, 13, 14, 15, 17, 18 (only amplitude and period), MC 1&2