Lesson 8: Applying the Double Angle Identities

The Double Angle identities are used when working with angles that are related to other angles by a multiple of 2.

The Double Angle Identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ → We can prove using sum of angles identities.

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ → $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= (1-\sin^2\alpha) - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

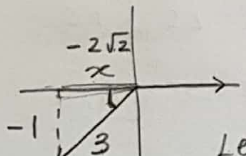
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \cos^2\alpha - (1-\cos^2\alpha)$

$= 2\cos^2\alpha - 1$

Example 1: Given angle θ is in standard position with its terminal arm in Quadrant 3 and $\sin\theta = -\frac{1}{3}$, determine the exact value of each trigonometric ratio.

a) $\sin 2\theta = 2\sin\theta\cos\theta$



$\sin\theta = \dfrac{-1}{3}$ ← opp / ← hyp

Let's calculate x,

$x^2 + y^2 = r^2$

$x^2 + (-1)^2 = 3^2$

$x^2 = 9 - 1$

$\sqrt{x^2} = \sqrt{8}$

$x = -\sqrt{8}$

$x = -\sqrt{4}\sqrt{2}$

$x = -2\sqrt{2}$

Therefore $\cos\theta = \dfrac{-2\sqrt{2}}{3}$ ← adj / ← hyp.

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(-\frac{1}{3}\right)\left(\frac{-2\sqrt{2}}{3}\right)$

$= \dfrac{4\sqrt{2}}{9}$

b) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

First, determine $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

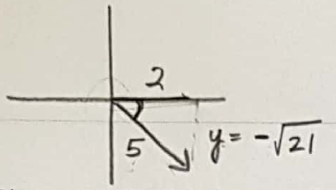
$= \dfrac{-\frac{1}{3}}{\frac{-2\sqrt{2}}{3}}$

$\tan\theta = \dfrac{1}{2\sqrt{2}}$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{\cancel{2}\left(\frac{1}{\cancel{2}\sqrt{2}}\right)}{1-\left(\frac{1}{2\sqrt{2}}\right)^2}$

$= \dfrac{\frac{1}{\sqrt{2}}}{1-\frac{1}{8}} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{8}{8}-\frac{1}{8}} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{7}{8}} = \dfrac{8}{7\sqrt{2}}$



$\cos\theta = \frac{2}{5}$ ← adj, ← hyp

$x^2 + y^2 = r^2$

$(2)^2 + y^2 = 5^2$

$\sqrt{y^2} = -\sqrt{21}$

Example 2: Given angle θ is in standard position with its terminal arm in Quadrant 4 and $\cos\theta = \frac{2}{5}$, determine the exact value of each trigonometric ratio.

$\sin\theta = \frac{-\sqrt{21}}{5}$ ← opp, ← hyp

a) $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{-\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)$

$= \frac{-4\sqrt{21}}{25}$

b) $\cos 2\theta = 2\cos^2\theta - 1$

$= 2(\cos\theta)^2 - 1$

$= 2\left(\frac{2}{5}\right)^2 - 1$

$= 2\left(\frac{4}{25}\right) - 1$

$= \frac{8}{25} - 1$

$= \frac{8}{25} - \frac{25}{25}$

$= \frac{-17}{25}$

Example 3: Write each expression as a single trigonometric ratio, then evaluate where possible.

a) $\sin\frac{\pi}{3}\cos\frac{\pi}{3}$

We know

$\frac{\sin 2\theta}{2} = \frac{2\sin\theta\cos\theta}{\cancel{2}}$

$= \frac{\sin 2\left(\frac{\pi}{3}\right)}{2}$

$= \frac{\sin\frac{2\pi}{3}}{2}$

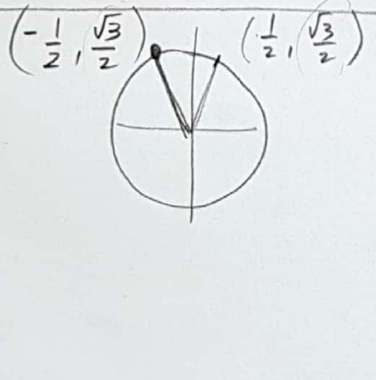
b) $6\cos^2\theta - 3$

$= 3(2\cos^2\theta - 1)$

$= 3(\cos 2\theta)$

* We cannot evaluate b/c we don't know what θ is.

$\frac{\sin\frac{2\pi}{3}}{2}$



$= \frac{\frac{\sqrt{3}}{2}}{\frac{2}{1}}$

$= \frac{\sqrt{3}}{4}$

OR $\boxed{\sin\frac{\pi}{3}\cos\frac{\pi}{3}} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4}$

Example 4: Prove the following identities.

a) $\cot x \csc 2x = \dfrac{1}{2\sin^2 x}$

b) $\dfrac{1-\cos 2x}{\sin 2x} \stackrel{?}{=} \tan x$

a)

LS	RS
$\left(\dfrac{\cos x}{\sin x}\right)\left(\dfrac{1}{\sin 2x}\right)$	
$= \left(\dfrac{\cancel{\cos x}}{\sin x}\right)\left(\dfrac{1}{2\sin x \cancel{\cos x}}\right)$	
$= \dfrac{1}{2\sin^2 x}$	$\dfrac{1}{2\sin^2 x}$
LS	RS

b)

LS	RS
$\dfrac{1-\cos 2x}{\sin 2x}$	$\tan x$
$= \dfrac{1-(1-2\sin^2 x)}{2\sin x \cos x}$	
$= \dfrac{\cancel{2}\sin^{\cancel{2}\,1} x}{\cancel{2}\cancel{\sin x}\cos x}$	
$= \dfrac{\sin x}{\cos x}$	
$= \tan x$	$\tan x$
LS	$=$ RS

OMIT NOT AN IDENTITY

~~c)~~ $\cos 2x = \cos^2 x - 2\tan^2 x$

sum/diff trig identity.

d) $\sin 5x \cos 3x - \cos 5x \sin 3x = 2\sin x \cos x$

$\sin 5x \cos 3x - \cos 5x \sin 3x$

$= \sin(5x - 3x)$

$= \sin 2x$

$= 2\sin x \cos x$

$2\sin x \cos x$

LS = RS

Example 5: Solve the following equations over the domain $0 \le x < 2\pi$.

a) $\cos 2x = 1 - 2\sin x$

$1 - 2\sin^2 x = 1 - 2\sin x$

$\cancel{1} - 2\sin^2 x - \cancel{1} + 2\sin x = 0$

$-2\sin^2 x + 2\sin x = 0$

$-2\sin x \underbrace{(\sin x - 1)} = 0$

$\sin x = 0$

$x = 0^r$

$\sin x - 1 = 0$

$\sin x = 1$

$x = \frac{\pi}{2}$

two solutions

$x = 0^r, \frac{\pi}{2}$

b) $\frac{1}{2}\sin 2x - \cos^2 x = 0$

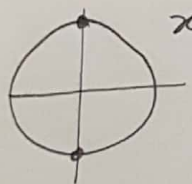
$\frac{1}{\cancel{2}}(\cancel{2}\sin x \cos x) - \cos^2 x = 0$

$\sin x \cos x - \cos^2 x = 0$

$\cos x \underbrace{(\sin x - \cos x)} = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$



$\sin x - \cos x = 0$

$\sin x = \cos x$

$\frac{\pi}{4}$ $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\frac{5\pi}{4}$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Another way of thinking about this,

$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Assignment Time! Work on: p.658- 5-18, MC 1&2

There are 4 solutions: $\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}}$