For $\cos x = 0, x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Verify by substitution that neither value of x is a root of the given equation.

The roots are: $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$

Verify the roots by substitution.

Here are some strategies to use to prove an identity:

- Start by simplifying the side of the identity that is more complex.
- Write the expressions in terms of $\sin x$ and $\cos x$.

Discuss the Ideas

1. What is the difference between a trigonometric identity and a trigonometric equation? Suppose you are given a trigonometric equation. What strategy can you use to check whether it might be an identity?

2. Can you conclude that an equation is an identity when it is shown to be valid for a given value of the variable? Explain.

Exercises

A

3. Write each expression in terms of a single trigonometric function.

a) $\frac{\cos \theta}{\sin \theta}$

$= \cot \theta$

b) $\frac{\sin^2 \theta}{\cos^2 \theta} = \left(\frac{\sin \theta}{\cos \theta}\right)^2$

$= \tan^2 \theta$

c) $\sin^2\theta \sec\theta \cos\theta \csc\theta$

$\sin^2\theta\left(\frac{1}{\cos\theta}\right)\cos\theta\left(\frac{1}{\sin\theta}\right)$

$= \sin\theta$

d) $\frac{\sin^2\theta}{\tan^2\theta} = \frac{\sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}}$

$= \left(\frac{\sin^2\theta}{1}\right)\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$

$= \cos^2\theta$

4. Determine the non-permissible values of θ.

a) $\sec\theta = \frac{1}{\cos\theta}$

$\cos\theta \neq 0$

$\theta = 90, 270°$

$90 + k(180°)\ k \in \mathbb{Z}$

b) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos\theta \neq 0$

$\theta \neq 90, 270,$

$\theta \neq \pm 90 + 180(k),\ k \in \mathbb{Z}$

c) $\frac{\csc\theta}{\cos\theta}$

$= \frac{\frac{1}{\sin\theta}}{\cos\theta}$

$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$

$= \frac{1}{\sin\theta(\cos\theta)}$

npv

$\sin\theta \neq 0$

$\theta = 0 + \pi(k)$

$k \in \mathbb{Z}$

OR

$\cos\theta \neq 0$

$\theta = \pm\frac{\pi}{2} + \pi(k)$

$k \in \mathbb{Z}$

d) $\frac{\sec\theta}{\sin\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta}$

$= \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}$

$= \frac{1}{\cos\theta \sin\theta}$

npv:

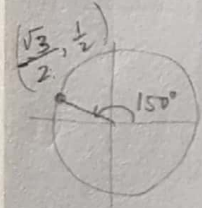
$\pi k,\ k \in \mathbb{Z}$

or

$\frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$

5. Verify each identity for the given value of θ.

a) $\tan\theta \csc\theta \sec\theta = \sec^2\theta$; $\theta = 150°$



$\tan 150 (\csc 150)(\sec 150)$

$\left(-\frac{1}{\sqrt{3}}\right)(2)\left(-\frac{2}{\sqrt{3}}\right)$

$= \frac{4}{3}$

$\sec^2(150)$

$= \left(\frac{2}{\sqrt{3}}\right)^2$

$= \frac{4}{3}$

b) $\frac{\tan\theta \csc^2\theta}{\sec^2\theta} = \cot\theta; \boxed{\theta = \frac{4\pi}{3}}$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$$\frac{\left(\tan\frac{4\pi}{3}\right)\left(\csc\frac{4\pi}{3}\right)^2}{\left(\sec\frac{4\pi}{3}\right)^2}$$

$$= \frac{(\sqrt{3})\left(\frac{2}{-\sqrt{3}}\right)^2}{\left(-\frac{2}{1}\right)^2}$$

$$= \frac{\sqrt{3}\left(\frac{4}{+3}\right)}{4}$$

$$= \frac{\cancel{4}\sqrt{3}}{3} \cdot \frac{1}{\cancel{4}}$$

$$= \frac{\sqrt{3}}{3}$$

$$= \frac{1}{\sqrt{3}}$$

$\cot\theta$

$\cot\left(\frac{4\pi}{3}\right)$

$$= \frac{\cos\frac{4\pi}{3}}{\sin\frac{4\pi}{3}}$$

$$= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}}$$

B

6. Prove each identity in question 5.

$$LS = \frac{\tan\theta\csc^2\theta}{\sec^2\theta}$$

$$= \frac{\frac{\cancel{\sin\theta}}{\cos\theta} \cdot \frac{1}{\sin^{\cancel{2}}\theta}}{\frac{1}{\cos^2\theta}}$$

$$= \frac{1}{\cancel{\cos\theta}\sin\theta} \cdot \frac{\cos^{\cancel{2}}\theta}{}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$= \cot\theta$

$= RS$

7. For each identity:

i) Verify the identity using graphing technology.

ii) Prove the identity.

a) $1 - \sin\theta = (\sin\theta)(\csc\theta - 1)$ b) $-\cot\theta = \dfrac{1 - \cot\theta}{1 - \tan\theta}$

$RS = \sin\theta\left(\frac{1}{\sin\theta} - 1\right)$

$= \frac{\sin\theta}{\sin\theta} - \sin\theta$

$= 1 - \sin\theta$

$= LS$

$RS = \dfrac{1 - \cot\theta}{1 - \tan\theta}$

$= \dfrac{1 - \frac{\cos\theta}{\sin\theta}}{1 - \frac{\sin\theta}{\cos\theta}}$

$= \dfrac{\frac{\sin\theta - \cos\theta}{\sin\theta}}{\frac{\cos\theta - \sin\theta}{\cos\theta}}$

$= \dfrac{\sin\theta - \cos\theta}{\sin\theta} \cdot \dfrac{\cos\theta}{\cos\theta - \sin\theta}$

$= \dfrac{-(\sin\theta + \cos\theta)}{\sin\theta} \cdot \dfrac{\cos\theta}{(\cos\theta - \sin\theta)}$

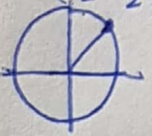
$= -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta = LS$

8. For each identity:

i) Verify the identity for $\theta = 45°$.

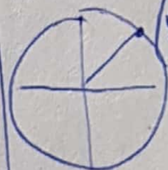
ii) Prove the identity.

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



a) $\dfrac{\cot\theta}{\cos\theta} - \csc\theta = 0$ b) $\tan^2\theta\cos^2\theta + \sin^2\theta = \dfrac{2}{\csc^2\theta}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



i) $\dfrac{\cot 45}{\cos 45} - \csc 45$ | 0

$= \dfrac{1}{\frac{\sqrt{2}}{2}} - \dfrac{2}{\sqrt{2}}$

$= \dfrac{2}{\sqrt{2}} - \dfrac{2}{\sqrt{2}}$

$= 0$ | $= 0$

ii)

LS	RS
$\dfrac{\cot\theta}{\cos\theta} - \csc\theta$	0
$= \dfrac{\frac{\cos\theta}{\sin\theta}}{\cos\theta} - \dfrac{1}{\sin\theta}$	
$= \dfrac{1}{\sin\theta} - \dfrac{1}{\sin\theta}$	
$= 0$	

$LS = RS$

i) $\tan^2 45\cos^2 45 + \sin^2 45$ | $\dfrac{2}{\csc^2 45}$

$= (1)^2\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$ | $= \dfrac{2}{\left(\frac{2}{\sqrt{2}}\right)^2}$

$= \frac{2}{4} + \frac{2}{4}$ | $= \dfrac{2}{\frac{4}{2}}$

$= 1$ | $= 1$

ii) $\tan^2\theta\cos^2\theta + \sin^2\theta$ | $\dfrac{2}{\csc^2\theta}$

$= \dfrac{\sin^2\theta}{\cancel{\cos^2\theta}}\cancel{\cos^2\theta} + \sin^2\theta$ | $= \dfrac{2}{\frac{1}{\sin^2\theta}}$

$= \sin^2\theta + \sin^2\theta$ | $= 2\left(\frac{\sin^2\theta}{1}\right)$

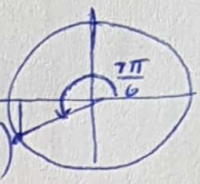
$= 2\sin^2\theta$ | $= 2\sin^2\theta$

$LS = RS$

9. For each identity:

i) Verify the identity for $\theta = \frac{7\pi}{6}$. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

ii) Prove the identity.



a) $\csc\theta = \dfrac{\csc\theta - 1}{1 - \sin\theta}$

i) $\csc\frac{7\pi}{6}$

$= -2$

$\dfrac{\csc\frac{7\pi}{6} - 1}{1 - \sin\frac{7\pi}{6}}$

$= \dfrac{-2 - 1}{\left(\frac{2}{2}\right)1 - -\frac{1}{2}}$

$= \dfrac{-3}{\frac{3}{2}}$

$= -\cancel{3}\left(\frac{2}{\cancel{3}}\right)$

$= -2$

LS = RS

ii) $\dfrac{\csc\theta - 1}{1 - \sin\theta}$ | $\csc\theta$

$= \dfrac{\frac{1}{\sin\theta} - 1\left(\frac{\sin\theta}{\sin\theta}\right)}{1 - \sin\theta}$

$= \dfrac{\frac{1 - \sin\theta}{\sin\theta}}{1 - \sin\theta}$

$= \dfrac{\cancel{(1 - \sin\theta)}}{\sin\theta} \cdot \dfrac{1}{\cancel{(1 - \sin\theta)}}$

$= \csc\theta$

LS = RS

b) $\dfrac{\cos\theta - \cot\theta}{1 - \sin\theta} = -\cot\theta$

i) $\dfrac{-\frac{\sqrt{3}}{2} - \sqrt{3}\left(\frac{2}{2}\right)}{\left(\frac{2}{2}\right)\frac{1}{1} - -\frac{1}{2}}$ | $-\cot\left(\frac{7\pi}{2}\right)$

$= -\sqrt{3}$

$= \dfrac{-\frac{3\sqrt{3}}{2}}{\frac{3}{2}}$

$= -\frac{\cancel{3}\sqrt{3}}{\cancel{2}}\left(\frac{\cancel{2}}{\cancel{3}}\right)$

$= -\sqrt{3}$

LS = RS

ii) $\dfrac{\cos\theta - \cot\theta}{1 - \sin\theta}$ | $-\cot\theta$

$\dfrac{\left(\frac{\sin\theta}{\sin\theta}\right)\cos\theta - \frac{\cos\theta}{\sin\theta}}{1 - \sin\theta}$

$= \dfrac{\frac{\sin\theta\cos\theta - \cos\theta}{\sin\theta}}{1 - \sin\theta}$

$= \dfrac{\frac{\cos\theta(\sin\theta - 1)}{\sin\theta}}{1 - \sin\theta}$

$= \dfrac{\frac{-\cos\theta\cancel{(1 - \sin\theta)}}{\sin\theta}}{\cancel{(1 - \sin\theta)}}$

$= -\cot\theta$ | $-\cot\theta$

10. Use algebra to solve each equation over the domain $0 \leq x < 2\pi$. Give the roots to the nearest hundredth where necessary.

$(\cos x - \sqrt{3}\sin x)$
multiply the conjugate

a) $\tan x = \cot x$

$\tan x = \dfrac{1}{\tan x}$

$\tan^2 x = 1$

$\tan^2 x - 1 = 0$

$(\tan x - 1)(\tan x + 1) = 0$

$\tan x + 1 = 0$ $\tan x - 1 = 0$

$\tan x = -1$ $\tan x = 1$

$x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

b) $\cos x + \sqrt{3}\sin x = 0$

$\cos^2 x - 3\sin^2 x = 0$

$\cos^2 x - 3(1 - \cos^2 x) = 0$

$\cos^2 x - 3 + 3\cos^2 x = 0$

$4\cos^2 x - 3 = 0$

$\sqrt{\cos^2 x} = \sqrt{\dfrac{3}{4}}$

$\cos x = \pm \dfrac{\sqrt{3}}{2}$

$\cos x = +\dfrac{\sqrt{3}}{2}$ $\cos x = -\dfrac{\sqrt{3}}{2}$

$x = \dfrac{\pi}{6}, \dfrac{11\pi}{6}$ $x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

c) $2\cos x = 7 - 3\sec x$

$2\cos x = 7 - 3\left(\dfrac{1}{\cos x}\right)$

$2\cos^2 x = 7\cos x - 3$

$2\cos^2 x - 7\cos x + 3 = 0$

$(2\cos x - 1)(1\cos x - 3) = 0$

$2\cos x - 1 = 0$ $\cos x = 3$

$\cos x = \dfrac{1}{2}$ No solution

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

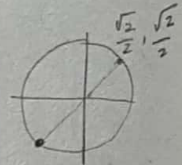
d) $\sin^2 x = \sin x \cos x$

$\sin^2 x - \sin x \cos x = 0$

$\sin x(\sin x - \cos x) = 0$

$\sin x = 0$ $\sin x - \cos x = 0$

$x = 0, \pi, 2\pi$ $\sin x = \cos x$



$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

11. Identify any errors in this proof, then write a correct algebraic proof.

To prove: $\dfrac{\sin\theta}{1 - \sin\theta} = \dfrac{1}{\csc\theta - 1}$

$$\begin{aligned}
\text{L.S.} &= \frac{\sin\theta}{1 - \sin\theta} \\
&= \frac{\sin\theta}{1} - \frac{\sin\theta}{\sin\theta} \\
&= \sin\theta - 1 \\
&= \frac{1}{\csc\theta} - 1 \\
&= \frac{1}{\csc\theta - 1} \\
&= \text{R.S.}
\end{aligned}$$

Start w/ complicated side

$$\begin{aligned}
RS &= \frac{1}{\csc\theta - 1} \\
&= \frac{1}{\frac{1}{\sin\theta} - 1\left(\frac{\sin\theta}{\sin\theta}\right)} \\
&= \frac{1}{\frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta}} \\
&= \frac{1}{\frac{1 - \sin\theta}{\sin\theta}} \\
&= \frac{\sin\theta}{1 - \sin\theta} = LS
\end{aligned}$$

12. Identify which equation below is an identity. Justify your answer. Prove the identity. Solve the other equation over the domain $-\pi \leq x \leq \pi$. Give the roots to the nearest hundredth.

a) $\frac{2\sin^2 x + 1}{\sin x} = 2\csc^2 x - 1$ b) $\frac{\sin^2 x + 1}{\sin x} = \frac{1 + \csc^2 x}{\csc x}$

LS	RS
$2\csc^2 x - 1$	$2\csc^2 x - 1$
$= \frac{2}{\sin^2 x} - 1$	
$= \frac{2}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}$	
$= \frac{2 - \sin^2 x}{\sin^2 x}$	
$= \frac{2}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}$	
$= 2\csc^2 x - 1$	

$\frac{\sin^2 x + 1}{\sin x}$

$\frac{1 + \csc^2 x}{\csc x}$

$= \frac{1 + \frac{1}{\sin^2 x}}{\frac{1}{\sin x}}$

$= \frac{\frac{\sin^2 x}{\sin^2 x} + \frac{1}{\sin^2 x}}{\frac{1}{\sin x}}$

$= \frac{\sin^2 x + 1}{\sin^2 x} \cdot \frac{\sin x}{1}$

$= \frac{\sin^2 x + 1}{\sin x}$

C

13. Here are two identities that involve the cotangent ratio:
$\cot\theta = \frac{1}{\tan\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

a) Show how you can derive one identity from the other.

$\cot\theta = \frac{1}{\tan\theta}$

$= \frac{1}{\frac{\sin\theta}{\cos\theta}}$

$= \frac{\cos\theta}{\sin\theta}$

b) Determine the non-permissible values of θ for each identity. Explain why these values are different. How could you illustrate this using graphing technology?

Multiple-Choice Questions

1. Which expression is the simplest form of $\sec \theta \tan \theta \cos \theta \csc \theta$?

A. $\sec \theta$ B. $\csc \theta$ C. $\frac{\cos \theta}{\sin^2 \theta}$ D. $\frac{\sin \theta}{\cos^2 \theta}$

$\sec\theta \tan\theta \cos\theta \csc\theta$

$\frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \cdot \cos\theta \cdot \frac{1}{\sin\theta}$

$= \sec \theta$

2. What are the restrictions on the expression $\frac{\sin \theta + 1}{\tan \theta}$?

denominator cannot equal zero

A. $\cos \theta \neq 0$

B. $\cos \theta \neq 0, \sin \theta \neq 0$

C. $\cos \theta \neq 0, \sin \theta \neq 0, \sin \theta \neq 1$

D. $\cos \theta \neq 0, \sin \theta \neq 1$

$\tan\theta = 0$

Where is $\tan \theta = 0$? and where is $\tan\theta$ undefined

$\frac{\sin\theta}{\cos\theta} = 0$

$\tan\theta = 0$ if $\sin\theta = 0$

$\tan\theta$ is undef if $\cos\theta = 0$