Exercises

A

3. For each expression below:

i) Determine any non-permissible values of θ.

ii) Write the expression as a single term.

a) $1 - \cos^2\theta$

$= \sin^2\theta$

b) $\cos^2\theta - 1$

$= -(1 - \cos^2\theta)$

$= -\sin^2\theta$

c) $\sec^2\theta - 1$

$= \frac{1}{\cos^2\theta} - 1\left(\frac{\cos^2\theta}{\cos^2\theta}\right)$

$= \frac{1-\cos^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

d) $1 - \sec^2\theta$

$= 1 - \frac{1}{\cos^2\theta}$

$= \frac{\cos^2\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta}$

$= \frac{-(1-\cos^2\theta)}{\cos^2\theta} = \frac{-\sin^2\theta}{\cos^2\theta} = -\tan^2\theta$

e) $\csc^2\theta - \cot^2\theta$

$= \frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}$

$= \frac{1-\cos^2\theta}{\sin^2\theta}$

$= \frac{\sin^2\theta}{\sin^2\theta}$

$= 1$

f) $\underbrace{\sin^2\theta + \cos^2\theta}_{} + 1$

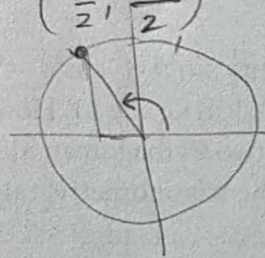
$= 1 + 1$

$= 2$

4. a) Verify the identity $\tan^2\theta + 1 = \sec^2\theta$ for $\theta = \frac{2\pi}{3}$.

$\cos\frac{2\pi}{3}$ $\sin\frac{2\pi}{3}$

$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$\tan^2\theta + 1$	$\sec^2\theta$
$\tan^2\left(\frac{2\pi}{3}\right) + 1$	$= \frac{1}{\left(-\frac{1}{2}\right)^2}$
$= (-\sqrt{3})^2 + 1$	$= \frac{1}{\frac{1}{4}}$
$= 3 + 1$	$= 4$
$= 4$	

b) Verify the identity $1 + \cot^2\theta = \csc^2\theta$ using graphing technology.

* use Desmos graphing calculator and graph

$y = 1 + \cot^2 x$ and $y = \csc^2 x$

We will find that they produce The SAME graph

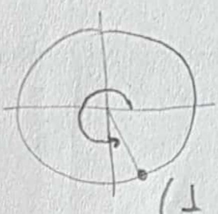
c) Verify the identity $\sin^2\theta + \cos^2\theta = 1$ for $\theta = 300°$.

$\sin^2 300 + \cos^2 300$

$= \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} + \frac{1}{4}$

$= \frac{4}{4} = 1$



$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

B

5. For each expression below:

i) Determine any non-permissible values of θ.

ii) Write the expression as a single term.

a) $\dfrac{\sqrt{1 - \sin^2\theta}}{\sqrt{1 + \tan^2\theta}}$

$= \dfrac{\sqrt{\cos^2\theta}}{\sqrt{\sec^2\theta}}$

$= \dfrac{\cos\theta}{\sec\theta}$

$= \dfrac{\cos\theta}{\frac{1}{\cos\theta}}$

$= \cos^2\theta$

b) $\dfrac{1 - \sin^2\theta + \cos^2\theta}{\cos\theta}$

$= \dfrac{\cos^2\theta + \cos^2\theta}{\cos\theta}$

$= \dfrac{2\cos^2\theta}{\cos\theta}$

$= 2\cos\theta$

c) $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{\cos\theta}{1 - \sin\theta}$

$= \dfrac{\cos\theta\,(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} + \dfrac{\cos\theta\,(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$= \dfrac{\cos\theta - \cos\theta\sin\theta}{1 - \sin^2\theta} + \dfrac{\cos\theta + \cos\theta\sin\theta}{1 - \sin^2\theta}$

$= \dfrac{\cos\theta - \cancel{\cos\theta\sin\theta} + \cos\theta + \cancel{\cos\theta\sin\theta}}{1 - \sin^2\theta}$

$= \dfrac{2\cos\theta}{\cos^2\theta}$

$= \dfrac{2}{\cos\theta}$

d) $\dfrac{\csc\theta}{\cot\theta + \tan\theta}$

$= \dfrac{\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$ ← make "like" denominators

$= \dfrac{\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right) + \frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}$

$= \dfrac{\frac{1}{\sin\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}}$

$= \dfrac{\frac{1}{\sin\theta}}{\frac{1}{\cos\theta\sin\theta}} = \frac{1}{\cancel{\sin\theta}}\left(\frac{\cos\theta\cancel{\sin\theta}}{1}\right) = \cos\theta$

6. For each identity:

i) Verify the identity using graphing technology.

ii) Prove the identity.

a) $1 - \cos^2\theta = \cos^2\theta \tan^2\theta$

$1-\cos^2\theta$	$\cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)$
	$= \sin^2\theta$
	$= 1-\cos^2\theta$

LS = RS

b) $\sin^2\theta + \cos^2\theta + \tan^2\theta = \sec^2\theta$

LS	RS
$\sin^2\theta + \cos^2\theta + \tan^2\theta$	$\sec^2\theta$
$= 1 + \tan^2\theta$	
$= 1 + \frac{\sin^2\theta}{\cos^2\theta}$	
$= \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$	
$= \frac{1}{\cos^2\theta}$	
$= \sec^2\theta$	

LS | RS

7. For each identity:

i) Verify the identity for $\theta = 240°$.

ii) Prove the identity.

a) $\cot^2\theta \sec\theta + \frac{1}{\cos\theta} = \csc^2\theta \sec\theta$

$\frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos\theta} + \frac{1}{\cos\theta}$

$= \frac{\cos\theta}{\sin^2\theta} + \frac{1}{\cos\theta}$

$= \frac{\cos\theta}{\sin^2\theta}\left(\frac{\cos\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\left(\frac{\sin^2\theta}{\sin^2\theta}\right)$

$= \frac{\cos^2\theta}{\sin^2\theta\cos\theta} + \frac{\sin^2\theta}{\cos\theta\sin^2\theta}$

$= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos\theta}$

$= \frac{1}{\sin^2\theta\cos\theta}$

$= \left(\frac{1}{\sin^2\theta}\right)\left(\frac{1}{\cos\theta}\right)$

$= \csc^2\theta\sec\theta$

b) $\frac{\tan\theta}{\cos\theta - \sec\theta} = -\csc\theta$

$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{1} - \frac{1}{\cos\theta}}$

$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta}{\cos\theta} - \frac{1}{\cos\theta}}$

$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta - 1}{\cos\theta}}$

$= \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \left(\frac{\cancel{\cos\theta}}{\cos^2\theta - 1}\right)$

$= \frac{\sin\theta}{\cos^2\theta - 1}$

$= \frac{\sin\theta}{-1(1-\cos^2\theta)}$

$= \frac{\sin\theta}{-1(\sin^2\theta)}$

$= \frac{1}{-1\sin\theta}$

$= -\csc\theta$

8. Is either of these statements true? Justify your answer.

a) Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$

yes $(\tan\theta)^2 = \left(\frac{\sin\theta}{\cos\theta}\right)^2$

$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$

b) Since $\sin^2\theta + \cos^2\theta = 1$, then $\sin\theta + \cos\theta = 1$ NO

9. For each identity:

i) Determine the non-permissible values of θ.

ii) Prove the identity.

a) $\frac{1}{\csc\theta + \cot\theta} = \csc\theta - \cot\theta$

$= \frac{1}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}}$

$= \frac{1}{\frac{1+\cos\theta}{\sin\theta}}$

$= \frac{\sin\theta}{(1+\cos\theta)} \frac{(1-\cos\theta)}{(1-\cos\theta)}$

$= \frac{\sin\theta - \sin\theta\cos\theta}{1-\cos^2\theta}$

$= \frac{\sin\theta - \sin\theta\cos\theta}{\sin^2\theta}$

$= \frac{\sin\theta}{\sin^2\theta} - \frac{\sin\theta\cos\theta}{\sin^2\theta}$

$= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$

$= \csc\theta - \cot\theta$

$=$ LS = RS

b) $\sin\theta + \frac{\cos\theta}{\tan\theta} = \frac{1}{\cos\theta\tan\theta}$

$= \sin\theta + \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}}$

$= \sin\theta + \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)$

$= \sin\theta + \frac{\cos^2\theta}{\sin\theta}$

$= \frac{\sin\theta}{1}\left(\frac{\sin\theta}{\sin\theta}\right) + \frac{\cos^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta}$

$\frac{1}{\cos\theta\tan\theta}$

$= \frac{1}{\cos\theta \cdot \left(\frac{\sin\theta}{\cos\theta}\right)}$

$= \frac{1}{\sin\theta}$

LS = RS

* Here we worked out and simplified both sides and we ended at the same place on each side of equation.

12. a) Prove this identity: $\frac{\cot\theta}{\csc\theta + 1} = \frac{\csc\theta - 1}{\cot\theta}$

LS RS

* We worked out each side and we stopped at the same expression.

LS

$$\frac{\cot\theta}{\csc\theta + 1}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta} + 1\left(\frac{\sin\theta}{\sin\theta}\right)}$$

$$= \frac{\frac{\cos}{\sin\theta}}{\frac{1+\sin\theta}{\sin\theta}}$$

$$= \frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta}{(1+\sin\theta)} \cdot \frac{\cos\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$

$$= \boxed{\frac{1-\sin\theta}{\cos\theta}}$$

RS

$$\frac{\csc\theta - 1}{\cot\theta}$$

$$= \frac{\frac{1}{\sin\theta} - 1\left(\frac{\sin\theta}{\sin\theta}\right)}{\frac{\cos\theta}{\sin\theta}}$$

$$= \frac{\frac{1-\sin\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$$

$$= \boxed{\frac{1-\sin\theta}{\cos\theta}}$$

b) Predict a similar identity involving $\tan\theta$ and $\sec\theta$. Prove this identity.

C

13. Determine a single trigonometric function for m such that the equation $\frac{2 - \sin^2\theta}{\cos\theta} = m + \cos\theta$ is an identity. Verify your answer by proving the identity.

VERIFY

$$\frac{2-\sin^2\theta}{\cos\theta}$$

$$= \frac{2-(1-\cos^2\theta)}{\cos\theta}$$

$$= \frac{1+\cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}$$

$$= \sec\theta + \cos\theta$$

Therefore $m = \sec\theta$

$$\sec\theta + \cos\theta$$

$$= \frac{1}{\cos\theta} + \frac{\cos\theta}{1}$$

$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}$$

$$= \frac{1+\cos^2\theta}{\cos\theta}$$

$$= \frac{1+(1-\sin^2\theta)}{\cos\theta}$$

$$= \frac{2-\sin^2\theta}{\cos\theta}$$

Multiple-Choice Questions

1. Which expression is equivalent to $\cos^2\theta + \sin^2\theta\cot^2\theta$?

A. $\cos^4\theta$ B. $2\cos^2\theta$ C. $\cot^2\theta$ D. $1 + \cot^2\theta$

$$\cos^2\theta + \sin^2\theta\cot^2\theta$$

$$= \cos^2\theta + \sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= 2\cos^2\theta$$

2. Which expression is equivalent to $\tan\theta + \cot\theta$?

A. 1 B. $\cos\theta\sin\theta$ C. $\csc\theta\sec\theta$ D. $\sin\theta + \cos\theta$

$$\tan\theta + \cot\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\cos\theta\sin\theta}$$

$$= \sec\theta\csc\theta$$